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SUFFICIENT CONDITIONS OF OPTIMALITY FOR A SURVIVAL PROBLEM†

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The problem of the containment of the trajectories of a differential inclusion in a specified closed set in a maximum time interval [1–4], which is a problem of optimal control with phase constraints, is considered. Sufficient conditions for optimality are obtained in the form of a maximum principle (cf. [5]). © 1997 Elsevier Science Ltd. All rights reserved.

1. The problem of the containment of the trajectories of the differential inclusion

$$\dot{x} \in F(x) \tag{1.1}$$

in a specified set G in a maximum time interval is considered. It is assumed that G is a non-empty closed subset of $\mathbb{R}^d$ and that $F(x)$ is a non-empty compact subset of $\mathbb{R}^d$ for each $x \in \mathbb{R}^d$.

A solution of inclusion (1.1) in a time interval I is understood to be an absolutely continuous function $x(\cdot): I \to \mathbb{R}^d$ such that $\dot{x}(t) \in F(x(t))$ almost everywhere in I.

The union of all solutions of inclusion (1.1) which satisfy the initial condition $x(0) = x_0$ is denoted by $Y(x_0)$.

The quality functional $T(x_0, x(\cdot)) = \sup\{t \geqslant 0 \mid x(r) \in G$ for all $r \in [0; t]$ is determined using the specified $x_0 \in G$ and $x(\cdot) \in Y(x_0)$.

The problem of survival in the domain G or, what is the same thing, the problem of avoiding encounters with the terminal set $M := \mathbb{R}^d G$ for a specified initial state $x_0 \in G$ is formulated in the following manner [1–4]

$$T(x_0, x(\cdot)) \to \sup, \quad x(\cdot) \in Y(x_0) \tag{1.2}$$

A trajectory $x(\cdot) \in Y(x_0)$ is said to be optimal in the case of an initial point $x_0 \in G$ if $T(x_0, x(\cdot)) = T(x_0)$, where

$$T(x_0) = \sup\{T(x_0, z(\cdot)) \mid z(\cdot) \in Y(x_0)\}$$

The completeness of the set G enables one to associate problem (1.2) with optimal control problems with phase constraints. Sufficient conditions for optimality in the case of problem (1.2) are presented below on the basis of results which have been previously obtained [5].

2. We introduce the many-valued mapping $Z(\cdot, \cdot): \mathbb{R}^d \times \mathbb{R}^d \to 2^{\mathbb{R}d}$, defined by the formula

$$Z(p, x) = \{z \mid (z, y - x) \leqslant C(F(x), p) - c(F(y), p) \quad \text{for all} \quad y \in G\},$$

where $c(X, \psi) = \sup\{(\psi, y) \mid y \in X\}$ is a support function of the set $X \subset \mathbb{R}^d$.

Definition. The function $p(\cdot)$: $[0, T] \to \mathbb{R}^d$ is called a conjugate function without a singularity for a trajectory $x(\cdot) \in Y(x_0)$ in the time interval $[0; T]$ if the function $p(\cdot)$ satisfies the following conditions: (a) it is continuous from the left, (b) it can be represented as the sum of an absolutely continuous function and a step function where all the points of the step τ_i, $i \in E(p(\cdot))$, $E(p(\cdot)) \subset \mathbb{N}$ lie in the interval $(0; T)$, and (c) the inclusion $\dot{p}(t) \in Z(p(t), x(t))$ holds almost everywhere in the interval $[0; T]$.

Theorem 1. Suppose that the quantity $T := T(x_0, x(\cdot))$ is finite for $x_0 \in G$ and $x(\cdot) \in Y(x_0)$. Next, suppose that a time $t_* \in (0; t]$ and a conjugate function $p(\cdot)$ without a singularity exist for $x(\cdot)$ in the interval $[0; T_*]$ such that:

1. the condition for a maximum

$$(\dot{x}(t), p(t)) = c(F(x(t)), p(t)) \tag{2.1}$$

is satisfied for almost all $t \in [0; t_*]$;

2. the step condition $(x(\tau_i), p^i) = c(G, p^i)$ is satisfied for all $i \in E(p(\cdot))$, where $p^i = p(\tau_i + 0) - p(\tau_i - 0)$;

3. the inequality $T(y_0) \leqslant T - t_*$ holds for all $y_0 \in G \cap \Pi(p(t_*), x(t_*))$, $\Pi(p, x) = \{x \mid (z - x, p) \leqslant 0\}$.

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Then, $x(\cdot)$ is the optimal trajectory of problem (1.2) for the initial point x_0.

Proof. Suppose that $x_0, x(\cdot), T, p(\cdot), t_*$ satisfy all the conditions of the theorem.
We assume that a $y(\cdot) \in Y(x_0)$ exists such that $T(x_0, y(\cdot)) > t_*$. Otherwise, the proof of the theorem is obvious. We consider the function $\xi(t) = (p(t), y(t) - x(t))$ in the interval $[0; t_*]$. Since $y(t) \in G$ for all $t \in [0; t_*]$, then, by making use of conditions 1 and 2 and the definition of a conjugate function, we have, by analogy with what has been described earlier [5]

$$\xi(t) = \int_0^t \dot{\xi}(r)dr + \sum_{i:\tau_i<t} [\xi(\tau_i+0) - \xi(\tau_i-0)] \leqslant \int_0^t \Phi(r)dr +$$

$$+ \sum_{i:\tau_i<t} [(y(\tau_i), p^i) - c(G, p^i)] \leqslant \int_0^t \Phi(r)dr \leqslant 0, \quad \Phi(r) = (\dot{y}(r), p(r)) - c(F(y(r)), p(r))$$

for all $t \in [0; t_*]$. It follows from this, in particular, that $y(t_*) \in G \cap \Pi(p(t_*), x(t_*))$. Consequently, according to condition 3, we have $T(x_0, y(\cdot)) \leqslant t_* + T(y(t_*)) \leqslant T(x_0, x(\cdot))$.
The theorem is proved.

Theorem 2. Suppose that the quantity $T := T(x_0, x(\cdot))$ is finite for $x_0 \in G$ and $x(\cdot) \in Y(x_0)$, and further suppose that a conjugate function $p(\cdot)$ without a singularity exists for $x(\cdot)$ in the interval $[0; T]$ such that:
1. $x(\cdot)$ is the unique solution of the inclusion (1.1) which belongs to $Y(x_0)$ and satisfied the condition for a maximum (2.1) almost everywhere in the interval $[0; T]$;
2. $(x(\tau_i), p^i) = c(G, p^i)$ for all $i \in E(p(\cdot))$;
3. $\operatorname{int} \Pi(p(T), x(T)) \subset M$.
Then, $x(\cdot)$ is the unique optimal trajectory of problem (1.2) corresponding to the initial point x_0.

Proof. Assume the opposite is true. Then, a trajectory $y(\cdot) \in Y(x_0)$ exists such that

$$T(x_0, y(\cdot)) \geqslant T(x_0, x(\cdot)) \tag{2.2}$$

where $y([0; T]) \neq x([0\ T])$.
As in the proof of Theorem 1, a function $\xi(t)$, $0 \leqslant t \leqslant T$ is considered and it is established that

$$\xi(T) \leqslant \int_0^T \Phi(r)dr$$

According to condition 1, we have from this that $\xi(T) < 0$. Consequently, by virtue of condition 3

$$y(T) \in \operatorname{int} \Pi(p(T), x(T)) \subset M$$

which contradicts (2.2). The theorem is proved.

3. *Example.* Suppose that $G = \{x \in \mathbb{R}^2 | (x, m) \geqslant 0\}$ and that the differential inclusion (1.1) is specified by the controlled system

$$\dot{x} = Ax + u, \quad x \in \mathbb{R}^2, \quad u \in P, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

where $m = (0, 1)'$, and P is an interval with vertices at the points $(1, 0)'$ and $(-2, -3)'$ (transposition is denoted by a prime).
Suppose that $x_0 = (-11/4, 4)$ and that $x(\cdot)$ is the solution of the Cauchy problem $x^{\cdot} = Ax + u(t), x(0) = x_0$ in which $u(t) = (-2, -3)'$ when $t \in [0; \pi) u(t) = (1/2\exp(t - \pi), 1/2\exp(t - \pi) - 1')$ when $t \in [\pi; \tau)$ and $u(t) = (1, 0)'$ when $\tau \leqslant t < +\infty$, where $\tau = \pi + \ln(4 - 2\sqrt{2})$. It can be verified that $T := T(x_0, x(\cdot)) = \tau + \pi/4$. Moreover, $x(t) \in \operatorname{int} G$ when $t \in [0; \pi) \cup (\tau; T)\ x(t) \in \operatorname{Fr} G$ in the interval $[\pi; \tau]$.
We put $p(t) = \exp((\pi - t)A)(-1, 1)'$ when $0 \leqslant t < \pi$ and $p(t) = \exp(\pi - t)(-1, 1)'$ when $\pi \leqslant t \leqslant \tau$.
It can be shown that all the conditions of Theorem 1 are satisfied with respect to $x_0, x(\cdot), p(\cdot)\ t_* = \tau$. In this case, the fact that condition 3 is satisfied can be checked using the alternating Pontryagin integral [6]. Hence, $x(\cdot)$ is the optimal trajectory for an initial point x_0.

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